Ex: Solve $y'' + y = 12\cos^2 x$

Sol: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = c_1 \cos x + c_2 \sin x$

$r(x) = 12\cos^2 x = 12\,\frac{1+\cos 2x}{2} = 6 + 6\cos 2x$

$\therefore y_p = c + A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x$

$y_p'' = -4A\cos 2x - 2B\sin 2x$

$\therefore -4A\cos 2x - 2B\sin 2x + A\cos 2x + B\sin 2x + C = 6 + 6\cos 2x$

$-3A\cos 2x - 3B\sin 2x + C = 6 + 6\cos 2x$

$\therefore$ **$C = 6$**, **$A = -2$**, **$B = 0$**

$\therefore y_p = 6 - 2\cos 2x \Rightarrow y = c_1\cos x + c_2\sin x + 6 - 2\cos 2x$

Ex: Solve $y'' - 4y' + 4y = 4e^{2x}$

Sol: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \Rightarrow y_h = c_1 e^{2x} + c_2 x e^{2x}$

$y_p = Ax^2 e^{2x}$

$y_p' = 2Ax^2 e^{2x} + 2Ax e^{2x}$

$y_p'' = 4Ax^2 e^{2x} + 4Ax e^{2x} + 4A e^{2x} + 4Ax e^{2x}$

$= 4Ax^2 e^{2x} + 8Ax e^{2x} + 2A e^{2x}$

$\therefore 4Ax^2 e^{2x} + 8Ax e^{2x} + 2A e^{2x} - 8Ax^2 e^{2x} - 8Ax e^{2x} + 4Ax^2 e^{2x} = 4e^{2x}$

$2A = 4 \Rightarrow$ **$A = 2$**

$\therefore y_p = 2x^2 e^{2x}$ and $y_h = c_1 e^{2x} + c_2 x^2 e^{2x}$

Ex: Solve $y'' - 4y' + 4y = 4e^{2x}$

Sol: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \Rightarrow$

$y_h = c_1 e^{2x} + c_2 x e^{2x}$

$\because r(x) = = 4e^{2x}$

$\therefore y_p = A x^2 e^{2x}$

$y_p' = 2Ax^2 e^{2x} + 2Ax e^{2x}$

$y_p'' = 4Ax^2 e^{2x} + 4Ax e^{2x} + 4A e^{2x} + 4Ax e^{2x}$

$= 4Ax^2 e^{2x} + 8Ax e^{2x} + 2A e^{2x}$

$=4Ax^2 e^{2x} + 8Ax e^{2x} + 2A e^{2x} - 4(2Ax^2 e^{2x} + 2Ax e^{2x}) + 4(A x^2 e^{2x}) = 4 e^{2x}$

$\therefore 4Ax^2 e^{2x} + 8Ax e^{2x} + 2A e^{2x} - 8Ax^2 e^{2x} - 8Ax e^{2x} + 4A x^2 e^{2x} = 4 e^{2x}$

$2A e^{2x} = 4 e^{2x}$

$2A = 4 \Rightarrow$ **$A = 2$**

$\because y_p = A x^2 e^{2x}$

$\therefore y_p = 2 x^2 e^{2x}$

$y = y_h + y_p$

$y = c_1 e^{2x} + c_2 x e^{2x} + 2 x^2 e^{2x}$

Ex: Solve $y'' - 9y = e^{3x} + \sin 3x$

Sol: $\lambda^2 - 9 = 0 \Rightarrow (\lambda-3)(\lambda+3) \Rightarrow \lambda_1 = 3, \lambda_2 = -3$

$y_h = c_1 e^{-3x} + c_2 e^{3x}$

$\because r(x) = e^{3x} + \sin 3x$

$\therefore y_p = e^{ax}(C_n x^n) + A \cos bx + B \sin bx$

$y_p = C x e^{3x} + A \cos 3x + B \sin 3x$

$y_p' = 3Cx e^{3x} + C e^{3x} - 3A \sin 3x + 3B \cos 3x$

$y_p'' = 9Cx e^{3x} + 3C e^{3x} + 3C e^{3x} - 9A \cos 3x - 9B \cos 3x$

$= 9C x e^{2x} + 6C x e^{2x} - 9A \cos 3x - 9B \sin 3x$

$\because y'' - 9y = e^{3x} + \sin 3x$

$\therefore 9Cx e^{3x} + 6C e^{3x} - 9A \cos 3x - 9B \sin 3x - 9x Ce^{3x} - 9A \cos 3x - 9B \sin 3x$

$= e^{3x} + \sin 3x$

$\because 6C e^{3x} = e^{3x}$ $\therefore C = 1/6$

$\because -9A \cos 3x - 9A \cos 3x = 0$ $\therefore A = 0$

$\because -9B \sin 3x - 9B \sin 3x = \sin 3x$ $\therefore B = -1/18$

$\because y_p = C x e^{3x} + A \cos 3x + B \sin 3x$

$\therefore y_p = 1/6 x e^{3x} - 1/18 \sin 3x$

$y = y_h + y_p$

$y = c_1 e^{-3x} + c_2 e^{3x} + 1/6\, x\, e^{3x} - 1/18 \sin 3x$

Ex: Solve $y'' + 4y = x \sin x, \quad y(0) = 0, \quad y'(0) = 1$

Sol: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow \alpha = 0, \beta = 2$

$y_h = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

$y_p = (A_1x + A_0) \cos x + (B_1x + B_0) \sin x$

$y_p' = -(A_1x + A_0) \sin x + A_1 \cos x + (B_1x + B_0) \cos x + B_1 \sin x$

$y_p'' = -(A_1x + A_0) \cos x - 2A_1 \sin x - (B_1x + B_0) \sin x + 2B_1 \cos x$

$y'' + 4y = x \sin x$

$\therefore 3(A_1x + A_0) \cos x + 3(B_1x + B_0) \sin x - 2A_1 \sin x + 2B_1 \cos x = x \sin x$

$A_1 = 0$, $B_0 = 0$, $A_0 = -2/9$, $B_1 = 1/3$

$y_p = (A_1x + A_0) \cos x + (B_1x + B_0) \sin x$

$\therefore y_p = -2/9 \cos x + 1/3\, x \sin x$

$y = c_1 \cos 2x + c_2 \sin 2x - 2/9 \cos x + 1/3\, x \sin x$

at x=0, y = 0,

$0 = c_1 + 0 - 2/9 + 0 \Rightarrow c_1 = \frac{2}{9}$

At $y'(0) = 1$

$$y' = -2c_1 \sin 2x + 2\, c_2 \cos 2x - \frac{2}{9}\sin x + \frac{1}{3} x \cos x + \frac{1}{3} \sin x$$

$x = 0, \quad y'(0) = 1$,

$$0 = c_1 + 0 - \frac{2}{9} + 0 + 0 + 0$$

$$c_2 = \frac{1}{2}$$

$\therefore$ The complete solution is $y = \frac{2}{9} \cos 2x + \frac{1}{2} \sin 2x - \frac{2}{9} \cos x + \frac{1}{3} x \sin x$

1- Variation of Parameter Method

Variation of parameter is a general method for find the particular solution $\mathbf{y_p}$ of linear ODE. Consider the following 2nd -order ODE: -

$$a_2\, y'' + a_2\;\, y' + a_0\, y = r(x)$$

The solution is

$$y(x) = y_h(x) + y_p(x)$$

Let

$$\boxed{y_h = c_1\, u_1(x) + c_2\, u_2(x)}$$

$$\boxed{y_p = v_1\,(x)\, u_1(x) + v_2(x)\, u_2(x)}$$

The practical solution $\boldsymbol{y_p}$ can be obtained by solving the following equations: -

$$v_1'\, u_1 + v_2'\, u_2 = 0$$

$$v_1'\, u'_1 + v_2'\, u'_2 = r(x)$$

$$\begin{bmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

$$\boxed{\Delta = u_1 u'_2 - u'_1 u_2}$$

$$v_1' = \frac{-u_2\, r(x)}{\Delta} \quad \Rightarrow \quad \boxed{v_1 = -\int \frac{u_2\, r(x)}{\Delta}\, dx}$$

$$v_2' = \frac{u_1\, r(x)}{\Delta} \quad \Rightarrow \quad \boxed{v_2 = \int \frac{u_1\, r(x)}{\Delta}\, dx}$$

Ex: Solve $y'' - 4y' + 3y = \frac{1}{1+e^{-x}}$

Sol: $\lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = 3 \quad , \lambda_2 = 1 \quad \Rightarrow \quad y_h = c_1 e^{3x} + c_2 e^{x}$

$y_h = c_1 u_1(x) + c_2 u_2(x)$

$\therefore u_1 = e^{3x} \quad \Rightarrow u_1' = 3e^{3x}$

$u_2 = e^{x} \quad \Rightarrow u_2' = e^{x}$

$\Delta = u_1 u'_2 - u'_1 u_2$

$\therefore \Delta = e^{3x}.e^{x} - 3e^{3x}.e^{x}$

$= -2e^{4x}$

$$v_1 = -\int \frac{u_2\, r(x)}{\Delta}\, dx$$

$$v_1 = -\int \frac{e^{x}.\frac{1}{1+e^{-x}}}{-2e^{4x}}\, dx$$

$$= \frac{1}{2}\int \frac{e^{x}.e^{-4x}}{1+e^{-x}}\, dx$$

$$= \frac{1}{2}\int \frac{e^{-3x}}{1+e^{-x}}\, dx$$

Let $z = 1 + e^{-x} \Rightarrow \quad e^{-x} = z - 1$

$dz = -e^{-x}\, dx \quad , \quad dx = \frac{dz}{-e^{-x}} \quad , \quad dx = -e^{x}\, dz$

$$= \frac{1}{2}\int \frac{e^{-3x}.-e^{x}}{z}\, dz$$

$$= \frac{1}{2}\int \frac{e^{-2x}}{z}\, dz \quad \Rightarrow \quad = \frac{1}{2}\int \frac{(e^{-x})^2}{z}\, dz \quad , e^{-x} = z - 1$$

$$\therefore v_1 = -\frac{1}{2}\int \frac{(z-1)^2}{z}\, dz$$

$$v_1 = -\frac{1}{2}\int \frac{z^2 - 2z + 1}{z}\, dz$$

$$= -\frac{1}{2}\int (z - 2 + \frac{1}{z})\, dz$$

$$= -\frac{1}{2}\,(\frac{z^2}{2} - 2\,z + \ln z)$$

$$v_{\ 1} = -\frac{1}{2}[\frac{1}{2}((1+e^{-x})^2 - 2(1+e^{-x}) + \ln(1+e^{-x})]$$

$$\because v_{\ 2} = \int \frac{u_1\, r(x)}{\Delta}\ dx$$

$$v_{\ 2} = \int \frac{e^{3x}\ \frac{1}{1+e^{-x}}}{-2\, e^{4x}}\ dx$$

$$v_{\ 2} = \frac{1}{2}\int -\frac{e^{3x}\ e^{-4x}}{1+e^{-x}}\ dx$$

$$= \frac{1}{2}\int \frac{-e^{-x}}{1+e^{-x}}\ dx$$

$$\frac{1}{2}\ln(1+e^{-x})$$

$$\therefore\ \ y_p = v_1\ u_1 + v_2\, u_2$$

$$\frac{-e^{3x}}{2}[\frac{1}{2}(1+e^{-x})^2 - 2(1+e^{-x}) + \ln(1+e^{-x})] + \frac{1}{2}e^{x}\ln(1+e^{-x})$$

$$y = \ y_h + \ y_p$$

Ex: Solve $\boldsymbol{y'' + y \ = \ tan\, x} \quad \boldsymbol{, y(0) = 1,\ \ y'(0) = 2}$

Sol: $\lambda^2 + 1 = 0$, ($\lambda = \pm i$) $\quad \alpha = 0\,, \beta = 1$

$$y_h \ = \ e^{\alpha x}(c_1 cos\beta x + c_2\ sin\beta x)$$

$$y_h = \ \ c_1 \cos x + \ \ c_2 \sin x$$

$$y_h = c_1\, u_1(x) + c_2\, u_2(x)$$

$$u_1 = \cos x \Rightarrow\ u'_1 = -\sin x$$

$$u_2 = \sin x \Rightarrow\ u'_2 = cosx$$

$$\Delta = u_1 u'_2 - u'_1 u_2$$

$\Delta = \cos x . \cos x - (-\sin x) \sin x = 1$

$v_1 = -\int \frac{u_2\, r(x)}{\Delta} \, dx$

$v_1 = -\int \frac{\sin x . \tan x}{1} \, dx = -\int \frac{\sin^2 x}{\cos x} \, dx \qquad ,\because \sin^2 x = 1 - \cos^2 x$

$= -\int \frac{1 - \cos^2 x}{\cos x} \, dx$

$-\int \frac{1}{\cos x} \, dx - \int \frac{\cos^2 x}{\cos x} dx \Rightarrow = -\int \sec x \, dx + \int \cos x \, dx$

$v_1 = -\ln|\sec x + \tan x| + \sin x$

$v_2 = \int \frac{u_1\, r(x)}{\Delta} \, dx$

$v_2 = \int \frac{\cos x \tan x}{1} \, dx = \int \sin x \, dx = -\cos x$

$y_p = v_1(x)\, u_1(x) + v_2(x)\, u_2(x)$

$\therefore \quad y_p = \cos x \,(-\ln|\sec x + \tan x) + \sin x) + \sin x\,(-\cos x)$

$= -\sin x \cos x - \cos x\,[\ln|\sec x + \tan x] + \sin x \cos x$

$y_p = -\cos x\,[\ln|\sec x + \tan x]$

$y = y_h + y_p$

$\therefore \quad y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|]$

$\cos 0 = 1, \sin 0 = 0$

y=1 at x=0 $\Rightarrow 1 = c_1 + 0 - 0 \Rightarrow c_1 = 1$

$y' = 2 \; at \; x = 0$

$y' = -c_1 \sin x + c_2 \cos x - \cos x . \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} + \sin x \ln|\sec x + \tan x|$

$2 = 0 + c_2 - 1 \Rightarrow \quad c_2 = 3$

$y = y_h + y_p$

$\therefore \; y = \cos x + 3 \sin x - \cos x \ln|\sec x + \tan x|$

2- Second order O.D.Es

- $y'' + 3y' + 2y = xe^{-x}$ ans: $y = Ae^{-x} + Be^{-2x} + xe^{-x}\left(\frac{1}{2}x - 1\right)$
- $y'' - 3y' + 2y = -4e^{x}\sinh x$ $y(0) = 2$, $y'(0) = 0$

Hint $\sinh x = \frac{e^{x} - e^{-x}}{2}$ ans: $e^{2x} - 2xe^{2x} + 1$

- $y'' - 2y' + y = \sin x + x^2$ ans $y = Ae^{x} + Bxe^{x} + \frac{1}{2}\cos x + x^2 + 4x + 6$
- $y'' + 2y' + 5y = 34\sin x\cos x$

ans: $y = e^{-x}(A\cos 2x + B\sin 2x) + \sin 2x - 4\cos 2x$

- $y'' + 9y = 2\cos(3x+4)$ $y = A\cos 3x + B\sin 3x + \frac{x}{3}\sin(3x+4)$

- $y'' + y = \csc x$ ans: $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\ln(\sin x)$
- $y'' + y = \sec x\tan x$

ans: $y = c_1\cos x + c_2\sin x + x\cos x - \sin x\ln(\cos x)$

- $y'' + 2y' + y = e^{-x}\ln x$

ans: $y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$

- $y'' - 4y' + 3y = x + e^{2x}$

ans: $y = c_1e^{x} + c_2e^{3x} + \frac{4}{9} + \frac{1}{3}x - e^{2x}$

- $y'' - 9y = e^{3x} + e^{-3x}$

ans: $y = c_1e^{3x} + c_2e^{-3x} + \frac{1}{6}xe^{3x} - \frac{1}{6}xe^{-3x}$

- $y'' - 2y' + y = e^{x}\sin x$

ans: $y = c_1e^{x} + c_2xe^{x} - e^{x}\sin x$

- $y'' - 16y = 19.2e^{4x} + 60e^{x}$ $y(0) = y'(0) = 0$

ans: $y = 2.2e^{4x} + 1.8e^{-4x} + 2.4xe^{4x} - 4e^{x}$