Formula of Special Functions

Non -elementary (special) function can be classified into:

1- Function defined by integral which can't be solved in term of elementary functions, like Gama, Beta, Error, and Q-function.

2- Function described by linear ODE such as Bessel, Legendre, and Green functions.

Gamma Function Γ(x):

Gamma Function is defined by the following integral:

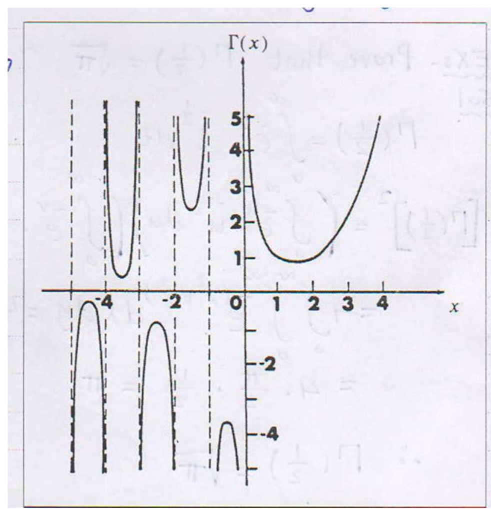
$$\Gamma(\mathrm{x}) = \int_0^\infty e^{-t} . t^{x-1}\, dt \qquad x > 0$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt \quad = 1$$

$$\Gamma(2) = \int_0^\infty t . e^{-t}\, dt \quad = 1$$

$$\Gamma(3) = \int_0^\infty t^2 . e^{-t}\, dt \quad = 2 = 2!$$

$$\Gamma(4) = \int_0^\infty t^3 . e^{-t}\, dt \quad = 6 = 3!$$



$$\therefore \Gamma(\mathrm{n}+1) = n\,\Gamma(\mathrm{n}) = n! \qquad n = positive\ integer$$

$$\Gamma(\mathrm{n}) = \infty \qquad \infty = negative\ integer$$

Integration by part gives the important functional relation of Gamma function as follows:

$$\Gamma(\mathrm{x}) = \int_0^\infty e^{-t} . t^{x-1}\, dt \quad \text{by part}$$

Let

$$\begin{Bmatrix} u = e^{-t} & \Rightarrow & du = -\,e^{-t} \\ dv = t^{x-1}\ dt & \Rightarrow & v = \dfrac{t^x}{x} \end{Bmatrix}$$

$$\Gamma(\mathrm{x}) = e^{-t} \; \frac{t^x}{x} - \int_0^\infty \frac{-e^{-t}.\; t^x}{x}\, dt$$

$$\Gamma(\mathrm{x}) = e^{-t} \; \frac{t^x}{x}\Big|_0^\infty + \frac{1}{x}\int_0^\infty e^{-t}.\; t^x \; dt \qquad \Gamma(\mathrm{x}) = 0 + \int_0^\infty e^{-t}\,.\,t^{x-1}\, dt$$

$$\because \; \Gamma(\mathrm{x}) = \int_0^\infty e^{-t}\,.\,t^{x-1}\, dt$$

$$\therefore \Gamma(\mathrm{x}+1) = \int_0^\infty e^{-t}\,.\,t^{(x+1)-1}\, dt$$

$$\therefore \Gamma(\mathrm{x}) = \frac{1}{x} \quad \Gamma(\mathrm{x}+1) \Rightarrow \frac{\Gamma(\mathrm{x}+1)}{x} \quad \Rightarrow$$

$$\therefore \boldsymbol{\Gamma(\mathrm{x}+1)) = x\, \Gamma(\mathrm{x})}$$

So that ***recursive property*** is

$$\Gamma(\mathrm{x}) = \frac{\Gamma(\mathrm{x}+1)}{x} = \frac{\Gamma(\mathrm{x}+2)}{x(x+1)} = \cdots \ldots\ldots = \frac{\Gamma(\mathrm{x}+\mathrm{k}+1)}{x(x+1)(x+2)\ldots\ldots(x+k)}$$

reflection formula is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi \; x} \qquad\qquad 0 < x < 1$$

EX: Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\,.\,t^{-\frac{1}{2}} \; dt$$

Let $t = y^2$, $dt = 2\,y\,dy$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2}\,.\,y^{-2\frac{1}{2}}\,.\,2\,y\,dy$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2}\,.\,y^{-1} \;\; 2\,y\,dy$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} \; dy$$

$$I = \int_0^{\infty} e^{-y^2}\ dy = \int_0^{\infty} e^{-x^2}\ dx$$

$$[\mathrm{I}]^2 = \left(\int_0^{\infty} e^{-y^2}\ dy)\ \right)\left(\int_0^{\infty} e^{-x^2}\ dx\right)$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} .dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-(r^2)} .r\,dr\ d\theta \quad *\ 2/2$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-(r^2)}\ .\ dr\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-(r^2)}\ .\ dr\ d\theta$$

$$I = 2.\frac{\sqrt{\pi}}{2} \qquad \therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

EX: Prove that $\Gamma(1.5), \Gamma(3.5), \Gamma(2.3)$

Sol:

$$\Gamma(\mathrm{x}+1)) = x\,\Gamma(\mathrm{x})$$

$$\Gamma(1.5) = \Gamma(0.5+1) = 0.5\ \Gamma(0.5) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(3.5) = 2.5\,\Gamma(2.5) \Rightarrow 2.5 * 1.5\,\Gamma(1.5) \Rightarrow 2.5 * 1.5 * 0.5\,\Gamma(1.5)$$

$$\therefore \Gamma(3.5) = 2.5 * 1.5 * 0.5\ \sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

$$\Gamma(2.3) = 1.3\,\Gamma(1.3) = 1.3 * 0.897471 = 1.1667123$$

Note there is a table of $\boldsymbol{\Gamma(x)}$ $\boldsymbol{for\ \ 1 \le x \le 2}$

Gama Function of **negative number** is calculated by the following formula:

$$\Gamma(\mathrm{x}) = \frac{\Gamma(\mathrm{x}+\mathrm{k}+1)}{x(x+1)(x+2)\ldots\ldots(x+k)} \begin{Bmatrix} x<0, & \Rightarrow & 1 \le x+k+1 \le 2 \\ & k = positive\ integer & \end{Bmatrix}$$

EX: Evaluate $\Gamma(-2.7)$?

Sol:

$$\Gamma(-2.7) = \frac{\Gamma(-2.7+3+1)}{-2.7\ (-2.7+1)(-2.7+2)(-2.7+3)} = \frac{\Gamma(1.3)}{-0.9639}$$

$$= -0.931082$$

Improper integrals can be solved in term of gamma function as illustrated in following examples:

EX: Evaluate $\int_0^\infty \sqrt{\mathrm{z}}\ .\ e^{-z^3}\ \ dz$

Sol:

Let $t = z^3$ $\quad$ *Changing the Limits* $\begin{Bmatrix} z=0, & \Rightarrow & t=0 \\ z=\infty & \Rightarrow & t=\infty \end{Bmatrix}$

$$\mathrm{z} = t^{\frac{1}{3}} \Rightarrow dz = \frac{1}{3}\ t^{-\frac{2}{3}}\,dt$$

$$\therefore \mathrm{I} = \int_0^\infty e^{-t}\ .\sqrt{t^{\frac{1}{3}}}\,.\frac{1}{3}\ t^{-\frac{2}{3}}\quad dt$$

$$\therefore \mathrm{I} = \int_0^\infty e^{-t}\ .\ t^{\frac{1}{6}}\ .\frac{1}{3}\ t^{-\frac{2}{3}}\quad dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t}\,.t^{-\frac{1}{2}}\ .dt$$

$$\because \Gamma(x) = \int_0^\infty e^{-t}\,.t^{x-1}\ .dt$$

$$\because \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} . t^{\frac{1}{2}-1} . dt$$

$$\frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{1}{3}\sqrt{\pi}$$

EX: Evaluate $I = \int_0^\infty \frac{x^c}{c^x} . \quad dx$

Sol: $I = \int_0^\infty x^c \; c^{-x} . \quad dx$

We can write c^{-x} *as* $e^{-x\ln c}$

$$= \int_0^\infty x^c \; e^{-x\ln c} . \quad dx$$

Let $t = x\ln c \Rightarrow dt = \ln c \, . dx \Rightarrow dx = \frac{dt}{\ln c}$

Substituting x in term of t:

$$x = \frac{t}{\ln c}$$

$$I = \int_0^\infty e^{-t} . \left(\frac{t}{\ln c}\right)^c . \frac{dt}{\ln c}$$

$$= \frac{1}{(\ln c)^{c+1}} \int_0^\infty e^{-t} . t^c . dt$$

$$= \frac{\Gamma(c+1)}{(\ln \;\;)^{c+1}}$$

Special Function Tutorial sheet

Gamma function

1- Find $\Gamma(5.2)$, $\Gamma(-4.6)$, $\Gamma(4.5)$, $\Gamma(-4.5)$

2- Evaluate the Function Integrals: -

a. $\int_0^{\infty} (x+1)^2 \ . e^{x^2} \ dx$

b. $\int_0^{\infty} \frac{-e^{\sqrt{x}}}{\sqrt{x}} \quad dx$

c. $\int_0^{1} [\ln(\left(\frac{1}{x}\right))]^{z-1} \quad dx$

d. $\int_0^{\infty} e^{-x^3} \quad dx$

Gamma function $\Gamma(\mathbf{x})$

Table Gamma function $\Gamma(\mathrm{x}) = \int_0^{\infty} e^{-t} \, . t^{x-1} \, dt \qquad x > 0$

	0.00	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
1.0	1.00000	0.984796	0.979849	0.975060	0.970425	0.965875	0.961521	0.957314	0.953249	0.949454
1.1	0.945682	0.942043	0.945682	0.934990	0.931732	0.928601	0.925787	0.922907	0.920146	0.917297
1.2	0.914764	0.912346	0.910258	0.908064	0.905979	0.903783	0.901909	0.900140	0.898695	0.897129
1.3	0.895663	0.894078	0.892810	0.891638	0.890774	0.889788	0.888687	0.887888	0.887180	0.886765
1.4	0.886233	0.885593	0.885239	0.884971	0.884979	0.884878	0.884681	0.884751	0.885082	0.885314
1.5	0.885460	0.885857	0.886337	0.887056	0.887691	0.888254	0.889050	0.890073	0.891022	0.891910
1.6	0.893017	0.894338	0.895596	0.896803	0.898217	0.899832	0.901395	0.902918	0.904752	0.906542
1.7	0.908301	0.910245	0.912369	0.914461	0.916531	0.918776	0.921191	0.923495	0.925967	0.928601
1.8	0.931221	0.933838	0.936693	0.939542	0.942393	0.945395	0.948549	0.951637	0.954871	0.958250
1.9	0.961576	0.965044	0.968652	0.972218	0.975923	0.979764	0.983627	0.987519	0.991596	0.995654